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Ascertaining complementary and incompatible quantum properties by means of double-slit experiments

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Abstract
The famous two-slits experiment is used to theoretically introduce the problem of detecting both which-slit (WS) property and another quantum property incompatible with the WS one, together with the measurement of the (complementary) position of the final impact point. General conditions for the existence of solutions are singled out, and a family of solutions is concretely found. Moreover, we theoretically design an ideal experiment which realizes this non-trivial detection.

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1. Introduction

Since the birth of quantum mechanics, the double-slit experiment showed its effectiveness in highlighting the conceptual puzzles of quantum theory [1–3], in particular in illustrating the duality between corpuscle-like and wave-like behavior of the physical entities. Here, it is used to introduce the following question: for each particle hitting and hence localized on the final screen, is it possible to ascertain which slit the particle passed through, but also another property *incompatible* with 'which slit' property?

Let us explain how this question enters the fundamental features of quantum theory, such as *complementarity* and *compatibility*[1] [4]. In a typical two-slits experiment, by *which-slit* (WS) property we mean the property stating which slit the particle is localized in, when it

[1] According to standard quantum theory [5] two observables are *compatible*, i.e. they can be measured together on the same individual specimen of the physical system, if they are represented by two self-adjoint operators A and B which commute with each other: $[A, B] = \mathbf{0}$. Hence, if $[A, B] \neq \mathbf{0}$ the two observables are incompatible and they cannot be measured together. The extreme case of incompatibility occurs when $[A, B] = ic$, i.e. when A and B are canonically conjugate. Following [4], in such a case the two observables turn out to be *complementary*. The *momentum* P and the *position* Q are the typical example of complementary observables: $[P, Q] = -i\hbar$

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crosses the slits' support. The WS property cannot be directly ascertained by means of the localization measurement it corresponds to, together with the measurement of the position of the impact on the final screen, because the two quantum observables corresponding to these measurements are complementary. For this reason, all devices conceived over the years to attain information about the WS property, such as the recoiling slit of Einstein [6], the light-electron scattering scheme of Feynman [7], the micro-maser apparatus of Englert, Scully and Walther (ESW) [8–10] (henceforth called *WS detectors*), yield *indirect* knowledge: they work by measuring a property T, *different* from the localization in one of the slits but compatible with the measurement of the final impact point, such that the slit taken by the particle can be inferred from the outcome of the measurement of T. Similar detectors, outflanking complementarity, can be devised also for properties other than the WS one. Then the question arises whether this kind of detection can be performed, on the same specimen of the physical system, for both such another property and WS property, in the case that they are *incompatible*, in the sense that the corresponding projection operators do not commute, together with the measurement of the final impact position.

In the present work this question, referred to as *problem* $(\mathcal{P})$, is investigated from a theoretical point of view. An ideal double-slit experiment is designed in which this non-trivial detection takes place. Moreover, we establish a method to find properties and state vectors which make possible this double detection.

It is worth to stress that the present work is not concerned with the investigation of the possible compromises between which-path knowledge and visibility of interference fringes, albeit provoked by erasure; this subject has thoroughly been studied and settled by several authors (see [11–13] and references therein). We seek for circumstances where which-path knowledge is ensured together with that of other incompatible properties, so that no erasure and hence no interference occurs.

Besides widening the picture of quantum complementarity and (non) compatibility, our investigation also addresses the problem of making inferences about *three* non-commuting observables. Though this issue was treated by Vaidman, Aharanov and Albert (VAA) (*how to ascertain the values of* σ_x, σ_y *and* σ_z *of a spin-1/2 particle*, [19]), our method yields inferences of a quite different nature with respect to those obtained by VAA.

In section 2 we formally introduce the concept of *WS detectors*. The more general notion of detectors is introduced in section 3, where problem $(\mathcal{P})$ is formulated in mathematical terms. In section 3.1 we present an ideal experiment, whose concrete realizability is discussed in section 3.2, which shows that our problem admits non-trivial solutions. In fact, to find this solution we have followed a mathematical method which we develop in section 4. Here we show that the existence of solutions of problem $(\mathcal{P})$ depends upon the dimension of the Hilbert space $\mathcal{H}_I$ the position of the particle is described in. If dim $(\mathcal{H}_I) < 4$, then no solution exists. If dim $(\mathcal{H}_I) = 4$, then for every solution the detections of the WS property and of the incompatible property are perfectly correlated (section 4).

In section 4.3 we prove that solutions without correlations exist if dim $(\mathcal{H}_I) \geqslant 6$. Moreover, a family of these solutions is concretely singled out.

In section 5 we outline how 'environment-induced decoherence' [14] relates to the present work; furthermore, some insights, from our results, into consistent histories theory [15–17] are introduced.

2. Which-slit detectors

Let us begin by introducing the quantum formalism for describing a two-slits experiment. We consider a localizable particle whose observable *position* is represented, at time t in the

Heisenberg picture, by an operator $Q^{(t)}$ of a suitable Hilbert space $\mathcal{H}_I$. Let the further degrees of freedom, related to spin or similar, be described in a second Hilbert space $\mathcal{H}_{II}$, in such a way that the complete Hilbert space is $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$. In general, if A_I (A_{II}) denotes a linear operator of $\mathcal{H}_I$ ($\mathcal{H}_{II}$), by the same symbol without index I (II) we denote the linear operator $A = A_I \otimes \mathbf{1}_{II}$ ($A = \mathbf{1}_I \otimes A_{II}$) acting on the whole space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$. Let us suppose that the Hamiltonian operator H of the entire system is essentially independent of the degrees of freedom described by $\mathcal{H}_{II}$, so that we may assume the ideal case $H = H_I \otimes \mathbf{1}_{II}$.

The projection operator identifying the WS property, 'the particle passes through slit 1', has the form $E = E_I \otimes \mathbf{1}_{II}$, where E_I is the projection operator which represents the property 'the particle is localized in slit 1 at time t_1', where t_1 is the time the particle crosses the screen supporting the slits. We may assume, without losing generality, that the property 'the particle passes through slit 2' is represented by $E'_I \otimes \mathbf{1}_{II}$, where $E'_I = \mathbf{1}_I - E_I$.

Given any interval Δ on the final screen, the event 'the particle hits Δ' coincides with the property 'the particle is localized in Δ at time t_2', where t_2 is the time of the final impact. Such a property is represented by the projection operator

$$F(\Delta) = \mathrm{e}^{\frac{i}{\hbar}H(t_2-t_1)} F^{(t_1)}(\Delta)\, \mathrm{e}^{-\frac{i}{\hbar}H(t_2-t_1)}, \tag{1}$$

where $F^{(t_1)}(\Delta)$ is the localization projection which represents the property 'the particle is in Δ at time t_1'. Hence, E and $F(\Delta)$ correspond to measurements of the position observables $Q^{(t_1)}$ and $Q^{(t_2)}$ respectively. Now, since between the times t_1 and t_2 the particle is interaction free, we have $H_I = \frac{P_I^2}{2m}$, and so $\frac{\mathrm{d}}{\mathrm{d}t}Q_I^{(t)} = \frac{i}{\hbar}\left[H_I, Q_I^{(t)}\right] = \frac{i}{2m\hbar}\left[P_I^2, Q_I^{(t)}\right] = \frac{P_I}{m}$. Then $Q_I^{(t_2)} = Q_I^{(t_1)} + \mathrm{i}\frac{P_I}{m}(t_2 - t_1)$ follows. Therefore E and $F(\Delta)$ turn out to be *complementary* observables, because $\left[Q_I^{(t_2)}, Q_I^{(t_1)}\right] = -\mathrm{i}\hbar\frac{t_2-t_1}{m}$. Thus, it is generally not possible to ascertain the WS property and the final impact point, directly by localization measurements.

However, if for a given state vector Ψ a projection operator of the kind $T = \mathbf{1}_I \otimes T_{II}$ exists such that equation $T\Psi = E\Psi$ holds, then it is possible *to detect* which slit each particle hitting the final screen passed through by means of a measurement of T. Indeed, since $[T, E] = \mathbf{0}$, the formula

$$p(T|E) = \frac{\langle\Psi|TE\Psi\rangle}{\langle\Psi|E\Psi\rangle}, \qquad \left(\text{respectively}\, p(E|T) = \frac{\langle\Psi|TE\Psi\rangle}{\langle\Psi|T\Psi\rangle}\right), \tag{2}$$

represents the probability that the outcome of T (respectively of E) is 1 if the outcome of E (respectively of T) is 1 [18]. It can be easily seen that equation $T\Psi = E\Psi$ is mathematically equivalent to state that both conditional probabilities in (2) are equal to 1, so that from the occurrence of outcome 1 (respectively 0) for T we can infer the passage of the particle through slit 1 (respectively 2). In other words, the condition $T\Psi = E\Psi$ entails an entanglement for Ψ that yields these correlations. Moreover, $F^{(t_1)}(\Delta)$ in (1) must have the form $F^{(t_1)}(\Delta) = F_I^{(t_1)}(\Delta) \otimes \mathbf{1}_{II}$, because it is a localization operator at time t_1, like E. Then, by using our assumption $H = H_I \otimes \mathbf{1}_{II}$ in (1) we find that $F(\Delta)$ must also have the form $F(\Delta) = F_I(\Delta) \otimes \mathbf{1}_{II}$, and this implies $[T, F(\Delta)] = \mathbf{0}$. Therefore, the measurement of T can be performed together with the final impact point and from its outcome we can infer the path taken by the particle; thus projection T can be used as a WS detector.

Example 1. The 'bullet' of the two-slits experiment proposed by ESW [8] is a *rubidium* atom in the long-lived excited state $63p_{3/2}$. Hilbert space $\mathcal{H}_{II}$ concerns with a pair of cavities $\hat{1}$ and $\hat{2}$ (see figure 1). The cavities are resonators for the electromagnetic field, tuned at a microwave frequency such that whenever the excited atom enters cavity $\hat{1}$ or $\hat{2}$, it decays emitting a photon. The event 'a photon is revealed in cavity $\hat{1}$ (respectively $\hat{2}$)' is represented by a projection operator $T_{II} = |1\rangle\langle 1|$ (respectively $T'_{II} = |2\rangle\langle 2|$) of $\mathcal{H}_{II}$. In this experimental situation the

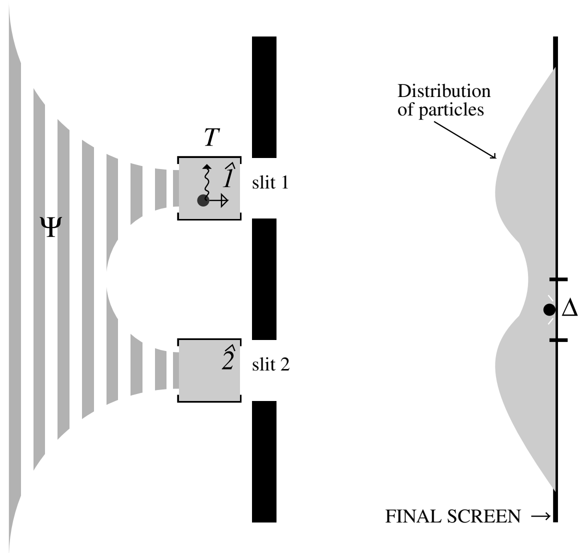

Figure 1. Which-slit detector.

complete state vector of the particle must have the form $\Psi = \frac{1}{\sqrt{2}}[\psi_1 \otimes |1\rangle + \psi_2 \otimes |2\rangle]$, where ψ_1 and $\psi_2 \in \mathcal{H}_I$ are state vectors respectively localized in slits 1 and 2 when the particle crosses the two-slits support, i.e. $E_I\psi_1 = \psi_1$, $E_I\psi_2 = 0$. A WS detector is represented by the projection operator $T = \mathbf{1}_I \otimes |1\rangle\langle 1|$; indeed $(\mathbf{1}_I \otimes |1\rangle\langle 1|)\Psi = (E_I \otimes \mathbf{1}_{II})\Psi$, i.e. $T\Psi = E\Psi$, and $[T, E] = \mathbf{0}$ trivially holds.

3. Detecting incompatible properties together

The concept of the WS detector can be extended to properties more general than the WS property by introducing the following definition.

Definition 1. *A projection operator Y of $\mathcal{H}$ is called a detector of a property $G = G_I \otimes \mathbf{1}_{II}$ with respect to the state vector Ψ if*
(i) $[Y, F(\Delta)] = \mathbf{0}$, (ii) $[Y, G] = \mathbf{0}$ and $Y\Psi = G\Psi$.

A measurement of Y detects G in exactly the same way a measurement of the WS detector T detects the WS property E. The following example turns out to be a detector for a property *incompatible* with the WS property, but it acts as a *quantum eraser* for the WS property.

Example 2. Let us go back to example 1. Once the vectors $|+\rangle = (1/\sqrt{2})(|1\rangle + |2\rangle) \in \mathcal{H}_{II}$ and $\psi_+ = 1/\sqrt{2}(\psi_1 + \psi_2) \in \mathcal{H}_I$ are defined, we introduce the projection operators $G_0 = |\psi_+\rangle\langle\psi_+| \otimes \mathbf{1}_{II}$ and $Y_0 = \mathbf{1}_I \otimes |+\rangle\langle +|$ respectively. Property G_0 can be detected on each particle hitting the final screen, because Y_0 turns out to be a detector of G_0, according to definition 1.

However, the two detectors Y_0 and T do not commute; therefore the detection of G_0 cannot be performed together with the detection of the WS property. Moreover, ESW argued [5] that if Y_0 is measured (and hence G_0 is detected) the information about the WS property

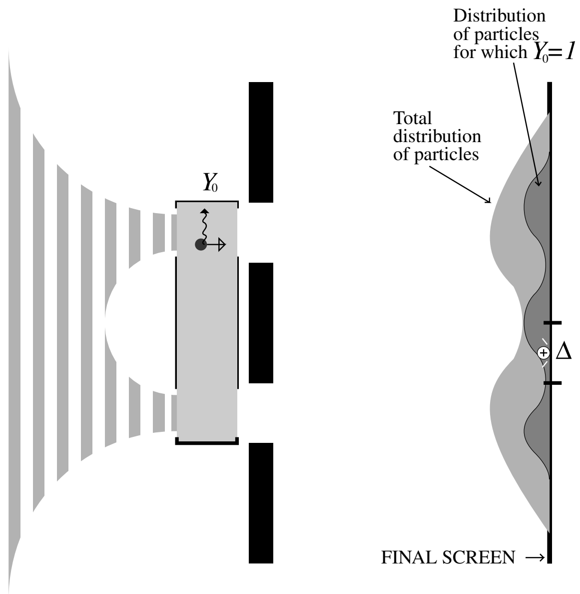

Figure 2. Erasure.

is definitively lost (*erased*), the occurrence of erasure being witnessed by the appearance of interference in the probability distribution of the particles impacted on the final screen for which $Y_0 = 1$ (figure 2).

In the present work we seek for the possibility of detecting, in the sense of definition 1, a property $G = G_I \otimes \mathbf{1}_{II}$ incompatible with the WS property E, *without erasing WS knowledge provided by a WS detector* T, for each particle hitting—and hence localized on—the final screen. To this aim, we require that with respect to the same state vector Ψ there exists both a WS detector T of E and a detector Y of G such that $[Y, T] = \mathbf{0}$, so that Y and T can be measured together, yielding detection of E and G. Condition $[Y, F(\Delta)] = \mathbf{0}$ will be automatically satisfied if Y has the form $Y = \mathbf{1}_I \otimes Y_{II}$. Therefore, our task will be successful if the following problem has a solution.

($\mathcal{P}$) Given the WS property $E = E_I \otimes \mathbf{1}_{II}$, we have to find a projection operator G_I of $\mathcal{H}_I$, two projection operators T_{II} and Y_{II} of $\mathcal{H}_{II}$ and a state vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$, such that the following conditions hold:

(C.1) $[E, G] \neq \mathbf{0}$, i.e $[E_I, G_I] \neq \mathbf{0}_I$;
(C.2) $[T, Y] = \mathbf{0}$, i.e $[T_{II}, Y_{II}] = \mathbf{0}_{II}$;
(C.3) $T\Psi = E\Psi$;
(C.4) $Y\Psi = G\Psi$;
(C.5) $0 \neq E\Psi \neq \Psi, 0 \neq G\Psi \neq \Psi$.

Condition (C.5) excludes the non-interesting solutions of (C.1)–(C.4), where Ψ is a simultaneous eigenvector of E or G. The following ideal experiment shows that this problem admits non-trivial solutions.

3.1. An ideal experiment

The bullet of our ideal experiment is a spin-3/2 particle whose position observable is described in a Hilbert space $\mathcal{H}_I$, while the spin observables are described in $\mathcal{H}_{II} \equiv \mathbf{C}^4$.

Let $\psi_1^{(1)}, \psi_1^{(2)}, \psi_1^{(3)}$ (respectively $\psi_2^{(1)}, \psi_2^{(2)}, \psi_2^{(3)}$) be three mutually orthonormal vectors of $\mathcal{H}_I$ localized in slit 1 (respectively slit 2) when the particle crosses the slits' support, i.e. such that $E_I\psi_1^{(k)} = \psi_1^{(k)}$ (respectively $E_I\psi_2^{(k)} = 0$). No further condition is required for these vectors. These six vectors form an orthonormal set. Then we take the Hilbert space $\mathcal{H}_I$ as the space generated by them. This implies that

$$E_I\varphi = \langle\psi_1^{(1)}|\varphi\rangle_I\psi_1^{(1)} + \langle\psi_1^{(2)}|\varphi\rangle_I\psi_1^{(2)} + \langle\psi_1^{(3)}|\varphi\rangle_I\psi_1^{(3)} \qquad \text{for every} \quad \varphi \in \mathcal{H}_I. \tag{3}$$

There are four eigenvectors $\alpha = |3/2\rangle, \beta = |1/2\rangle, \gamma = |-1/2\rangle, \delta = |-3/2\rangle \in \mathcal{H}_{II}$ corresponding to the four possible values (in $\hbar$ units) of the spin along direction z, represented by the Hermitian operator S_z of $\mathbf{C}^4$.

Let the particle be prepared in the entangled state (see section 3.2) represented by

$$\Psi = \frac{\sqrt{3}}{4}\left(\psi_1^{(1)} + \psi_1^{(2)}\right)|1/2\rangle + \frac{1}{\sqrt{8}}\psi_1^{(3)}|3/2\rangle + \frac{1}{4}\left(\psi_2^{(1)} + \psi_2^{(2)}\right)|-3/2\rangle + \sqrt{\frac{3}{8}}\psi_2^{(3)}|-1/2\rangle. \tag{4}$$

The four projection operators $A_{II} = |3/2\rangle\langle 3/2|, B_{II} = |1/2\rangle\langle 1/2|, C_{II} = |-1/2\rangle\langle -1/2|, D_{II} = |-3/2\rangle\langle -3/2|$ represent spin observables pertaining $\mathcal{H}_{II}$ (if $A = \mathbf{1}_I \otimes A_{II}$ has outcome 1, then the particle has spin-3/2 along z, and so on). They trivially commute with both $F(\Delta)$ and E. Then the projection operator $T = A + B = \mathbf{1}_I \otimes (|3/2\rangle\langle 3/2| + |1/2\rangle\langle 1/2|)$ also commute with $F(\Delta)$ and E. Now, by using (3) and (4) we obtain

$$E\Psi = \frac{\sqrt{3}}{4}\left(\psi_1^{(1)} + \psi_1^{(2)}\right)|1/2\rangle + \frac{1}{\sqrt{8}}\psi_1^{(3)}|1/2\rangle = T\Psi.$$

Therefore, T turns out to be a WS detector.

Now we introduce a property $G = G_I \otimes \mathbf{1}_{II}$ incompatible with E, which can be detected by means of a suitable detector Y without renouncing to the WS knowledge provided by T. Given any $\varphi \in \mathcal{H}_I$, we define

$$G_I\varphi = \langle\psi'|\varphi\rangle_I\psi' + \langle\psi''|\varphi\rangle_I\psi'' + \langle\psi'''|\varphi\rangle_I\psi''', \tag{5}$$

where $\psi' = 1/2\left(\psi_1^{(1)} - \psi_1^{(2)} + \psi_2^{(1)} - \psi_2^{(2)}\right)$, $\psi'' = \psi_1^{(3)}$, $\psi''' = \psi_2^{(3)}$.

A straightforward calculation based on (4) and (6) shows that

$$[G_I, E_I]\varphi = \frac{1}{4}\left\{\langle\psi_2^{(1)} - \psi_2^{(2)}|\varphi\rangle\psi_1^{(1)} - \langle\psi_2^{(1)} - \psi_2^{(2)}|\varphi\rangle\psi_1^{(2)} - \langle\psi_1^{(1)} - \psi_1^{(2)}|\varphi\rangle\psi_2^{(1)} \right.$$
$$\left. + \langle\psi_1^{(1)} - \psi_1^{(2)}|\varphi\rangle\psi_2^{(2)}\right\};$$

hence $[G, E] \neq \mathbf{0}$, so that G and E are incompatible with each other. However, the projection operator $Y = A + C = \mathbf{1}_I \otimes (|3/2\rangle\langle 3/2| + |-1/2\rangle\langle -1/2|)$ satisfies the condition $Y\Psi = G\Psi$ and it trivially commutes with $F(\Delta) = F_I(\Delta) \otimes \mathbf{1}_{II}$; therefore Y is a detector of G. Nevertheless, we have $[Y, T] = \mathbf{0}$; then Y and T can be both measured together with the position of the final impact. In other words, both properties G and E, mutually incompatible, can be detected together on each particle localized on the final screen. In the present work, aimed to theoretical investigations, we are not concerned with the question of the physical meaning of G, which evidently depends upon that of the vectors $\psi_k^{(1)}, \psi_k^{(2)}, \psi_k^{(3)}, k = 1, 2$. Thus, we have a solution of problem $(\mathcal{P})$.

The measurements of T and Y could be performed, as shown in figure 3, by measuring the observable S_z: the outcome 1 of T occurs when the outcome of S_z is 3/2 or 1/2 and, for the same specimen of the physical system, the outcome 1 of Y occurs if the outcome of S_z is 3/2 or $-1/2$. To avoid perturbation in the dynamics of the quantum system, such a spin measurement can be performed once the localization in Δ on the final screen took place (figure 3).

Such an ideal experiment allows us to make inferences, for each specimen of the physical system, about *all* three observables which are as follows:

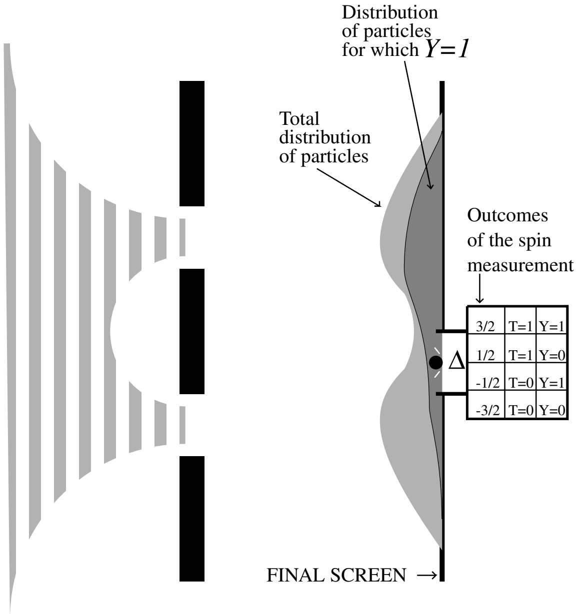

Figure 3. Ideal apparatus for detecting both E and G.

(1) the position of the final impact point, which is inferred from a direct measurement of $F(\Delta)$;
(2) the WS property E, whose value is inferred from the outcome of the WS detector T,
(3) the property G, which is inferred from the outcome of detector Y, like E from T.

This result is independent of the choice of the three orthonormal vectors $\psi_k^{(1)}, \psi_k^{(2)}, \psi_k^{(3)}, k = 1, 2$, within the sub-space they span, which determine the projection operator G, through (5). These vectors can then be chosen in such a way that G turns out to be incompatible (or even complementary) with $F(\Delta)$, i.e. such that $[G, F(\Delta)] \neq \mathbf{0}$; in such a case, the inferences refer to three *non-commuting* observables.

Another method to ascertain the values of three non-commuting observables, namely the three spin observables σ_1, σ_2 and σ_3 of a spin-1/2 particle, was devised by Vaidman *et al* in [19]. They proposed an ideal experiment in which a spin-1/2 particle is coupled with another 'external' particle, in such a way that the compound system is in a suitable known entangled state. During any run i of the experiment, just one, say $\sigma_{k(i)}$, of the three non-commuting observables $\sigma_1, \sigma_2, \sigma_3$, is actually measured, by means of an apparatus which leaves the entire system in an eigenstate of $\sigma_{k(i)}$ corresponding to the eigenvalue equal to the measured value. After such a spin measurement, a suitable observable A is measured, having the property that the outcome of the measured spin can be inferred from the outcome of A, *without knowing* which spin had been previously measured; i.e. the method yields inferences such as follows: if σ_1 has been measured the outcome is $+1/2$, if σ_2 has been measured the outcome is $-1/2$ and if σ_3 has been measured the outcome is $1/2$.

Here we want to remark the difference between the meaning of our result with respect to that obtained by VAA [19]. According to the latter, from the outcome $a(i)$ of A, the actual outcome of $\sigma_{k(i)}$ is retrodicted, while the inferences about the remaining two spins have a counterfactual character: 'if σ_j was measured instead of $\sigma_{k(i)}$ in the run i, and if the outcome of A had been $a(i)$, then its outcome would have been—for instance—$1/2$. These remaining inferences are not interpretable as detections, such as those used to ascertain the WS property, because $[A, \sigma_k] \neq \mathbf{0}$ forbids us to introduce the corresponding conditional probabilities such as (2). Furthermore, VAA's inferences can be drawn only under the hypothesis that the spin measurement actually performed leaves the system in an eigenstate of $\sigma_{k(i)}$.

According to our method, the value of one of the three non-commuting observables, $F(\Delta)$, is the outcome of an actually performed measurement, while the remaining two are inferred from actually performed detections of the same kind used in two-slits experiments to ascertain the WS property.

Remark. The general solution of the problem of ascertaining the values of two non-commuting projection operators, F and E with ranges $\mathcal{M}$ and $\mathcal{N}$, by measuring F and detecting E in the sense of definition 1, was given in [18]. In the case that the state vector ψ is such that $[F, E]\psi = 0$, i.e. if $\psi \in \mathcal{C}(F, E)$, where $\mathcal{C}(F, E) = (\mathcal{M} \cap \mathcal{N}) \otimes (\mathcal{M}^{\perp} \cap \mathcal{N}) \otimes (\mathcal{M} \cap \mathcal{N}^{\perp}) \otimes (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})$ is the so-called *commutation sub-space* of F and E, the solution singled out in [18] of this simpler problem turns out to be rather straightforward. Our problem $(\mathcal{P})$ is complicated by the requirement of detecting *two* non-commuting projection operators, E and G, besides measuring $F(\Delta)$. In the present case, we cannot assume either $\Psi \in \mathcal{C}(F(\Delta), E)$, $\Psi \in \mathcal{C}(F(\Delta), G)$ or $\Psi \in \mathcal{C}(F(\Delta), E, G)$, while $\Psi \in \mathcal{C}(E, G)$ holds as a consequence of (C.2)–(C.4).

3.2. Experimental issues

In the perspective of a realization of the double detection of E and G, a crucial experimental task is to create the entanglement, encoded in the state vector Ψ in (4), between the particle and the detector, before the time t_1 when the particle reaches the screen supporting the slits. This can be (ideally) realized in two steps. In the first step only particles with the x component of the spin equal to $3/2$ are selected, for instance by means of a suitable Stern–Gerlach apparatus. Hence, in the Schroedinger picture, at this stage—time $t_0 < t_1$—the state vector is of the kind $\psi|s\rangle$, with $\psi \in \mathcal{H}_I$, $\|\psi\| = 1$, and $S_x|s\rangle = 3/2|s\rangle$, so that we can take $|s\rangle = \frac{1}{\sqrt{8}}\{|3/2\rangle + \sqrt{3}|1/2\rangle + \sqrt{3}|-1/2\rangle + |-3/2\rangle\}$.

In the second step, during their flight between times t_0 and t_1, the particles undergo the action of another Stern–Gerlach magnet, able to deflect the particles with respect to S_z: the particles with $S_z = 3/2$ or $1/2$ (respectively $-1/2$ or $-3/2$) are forced to travel toward slit 1 (respectively 2), but through two alternative spatial channels according to the value, $3/2$ or $1/2$, of S_z, so that the dynamical evolution between times t_0 and t_1 is represented by a unitary operator U such that

$U(\psi|3/2\rangle) = \psi_1^{[\frac{3}{2}]}|3/2\rangle$, $U(\psi|1/2\rangle) = \psi_1^{[\frac{1}{2}]}|1/2\rangle$, $U(\psi|-3/2\rangle) = \psi_1^{[-\frac{3}{2}]}|-3/2\rangle$, $U(\psi|-1/2\rangle) = \psi_1^{[-\frac{1}{2}]}|-1/2\rangle$, where $\psi_k^{[J]}$ are vectors of $\mathcal{H}_I$ representing the alternative spatial channels taken by the particles to reach slit k; hence $\langle\psi_{k_1}^{[J_1]}|\psi_{k_2}^{[J_2]}\rangle = \delta_{k_1,k_2} \cdot \delta_{J_1,J_2}$ and $E_I\psi_1^J = \psi_1^{[J]}$, $E_I\psi_2^{[J]} = 0$. Now, if dim $(E_I\mathcal{H}_I)$, dim $((\mathbf{1}_I - E_I)\mathcal{H}_I) \geqslant 3$, then mutually orthonormal vectors $\psi_1^{\pm} \in E_I\mathcal{H}_I$, $\psi_2^{\pm} \in (\mathbf{1}_I - E_I)\mathcal{H}_I$ exist such that $\psi_1^{[\frac{1}{2}]} = \frac{1}{\sqrt{2}}(\psi_1^+ + \psi_2^-)$ and $\psi_2^{[-\frac{3}{2}]} = \frac{1}{\sqrt{2}}(\psi_2^+ + \psi_2^-)$. If we put $\psi_1^{(1)} = \psi_1^+$, $\psi_1^{(2)} = \psi_1^-$, $\psi_1^{(3)} = \psi_1^{[\frac{3}{2}]}$, $\psi_2^{(1)} = \psi_2^+$, $\psi_2^{(2)} = \psi_2^-$, $\psi_3^{(3)} = \psi_2^{[-\frac{1}{2}]}$, then the state vector Ψ outcoming from this dynamical preparing process must be $\Psi = U\left(\psi\frac{1}{\sqrt{8}}\{|3/2\rangle + \sqrt{3}|1/2\rangle + \sqrt{3}|-1/2\rangle + |-3/3\rangle\}\right) = \frac{\sqrt{3}}{4}(\psi_1^{(1)} + \psi_1^{(2)})|1/2\rangle + \frac{1}{\sqrt{8}}\psi_1^{(3)}|3/2\rangle + \frac{1}{4}(\psi_2^{(1)} + \psi_2^{(2)})|-3/2\rangle + \frac{\sqrt{3}}{8}\psi_2^{(3)}|-1/2\rangle$, which is just the state vector needed for our experiment.

However, it must be stressed the ideal character of our proposal. In fact, a real experiment, addressed to simultaneously detect both WS property and an incompatible one, is yet to be performed. In the literature, reports of actually realized experiments can be found in which the two detections are performed as mutually exclusive, or complementary, alternatives [20, 21]. We shall briefly compare our ideal experiment with these *realistic* experimental schemes.

In the experiment reported in [20] the traveling particle is a photon, and the role of the two slits is played by a Mach–Zehnder interferometer where the photon can take one of two alternative paths. These paths are entangled with the polarization $\Pi(\theta_0)$ of the photon in a given direction θ_0, so that the measurement of $\Pi(\theta_0)$ reveals which path (equivalent to which slit) has been taken by the photon. Hence in this experiment we have $\mathcal{H}_{II} = \mathbf{C}^2$, which is the proper Hilbert space for describing the photon polarization. In this situation the polarization $\Pi(\theta_1)$ along the direction $\theta_1 = \theta_0 + \pi/4$ is entangled with a property different from, and incompatible with, which-path property. Of course, this second detection is alternative to, and excludes, the first one ($[\Pi(\theta_0), \Pi(\theta_1)] \neq \mathbf{0}$). In our experimental scheme, the existence of the two simultaneous detectors Y and T for the two incompatible properties E and G requires that dim $(\mathcal{H}_{II}) \geqslant 4$, because four non-trivial mutually orthogonal projections $A_{II}, B_{II}, C_{II}, D_{II}$ must exist. Thus, this kind of realistic scheme cannot be adapted to realize an equivalent version of our experiment.

The experiment realized by Dürr *et al* [21] is quite different. The bullet is a *rubidium* atom (^{85}Rb) which interacts, *via* a Bragg-scattering process [22], with a standing electromagnetic wave playing the role of the two slits, and as a consequence it can take one of two alternative paths. In general, two separated interference patterns are produced by the atoms in the far field, and which-path information is not available. But with a suitable tuning of the standing wave each path entangles with one of two internal electronic states of the hyperfine structure of the atom, and which-path information can be obtained by measuring the hyperfine level of the atom. Hence also in this case dim $(\mathcal{H}_{II}) = 2$, while our scheme would require dim $(\mathcal{H}_{II}) \geqslant 4$. This suggests that this experimental method could eventually be adapted for an equivalent version of our experiment exploiting richer hyperfine structures.

4. Solving problem $\mathcal{P}$

In this section, we face the problem of mathematically finding solutions of problem ($\mathcal{P}$). In subsection A, by adopting a suitable matrix representation, we show that conditions C.1–C.4 impose some general constraints to the vector state Ψ and to the entries of the matrix which represents G.

In subsection B, we consider the case dim $(\mathcal{H}_I) \leqslant 4$. No solution exists if dim $(\mathcal{H}_I) < 4$. If dim $(\mathcal{H}_I) = 4$, then all solutions of ($\mathcal{P}$) are affected by a direct correlation between the detections of G and E.

In subsection C, the case dim $(\mathcal{H}_I) > 4$ is considered. It is proved that solutions without correlations exist if dim $(\mathcal{H}_I) \geqslant 6$. In particular, we show a method to concretely single out a family of these solutions.

4.1. General constraints

In a solution of problem ($\mathcal{P}$) condition (C.2) $[T_{II}, Y_{II}] = \mathbf{0}$ implies that four mutually orthogonal projection operators $A_{II}, B_{II}, C_{II}, D_{II}$ of $\mathcal{H}_{II}$ exist [23] such that

$$T_{II} = A_{II} + B_{II}, \qquad Y_{II} = A_{II} + C_{II} \qquad \text{and} \qquad A_{II} + B_{II} + C_{II} + D_{II} = \mathbf{1}_{II}. \tag{6}$$

Hilbert space $\mathcal{H}_I$ has an an orthonormal basis $\{f_1, f_2, ...\} \cup \{g_1, g_2, ...\}$ formed with eigenvectors of E_I such that $E_I f_i = f_i$, for all i and $E_I g_k = E_I g_k = 0$ for all k. Thereby, every state vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ can be uniquely decomposed as $\Psi = \sum_i f_i \otimes \mathbf{x}_i + \sum g_k \otimes \mathbf{y}_k$, where $\mathbf{x}_i, \mathbf{y}_k \in \mathcal{H}_{II}$. Then Ψ shall be represented as a column vector:

$$\Psi = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \cdot \\ \cdot \end{bmatrix} = [\underbrace{a_1, b_1, c_1, d_1}_{\mathbf{x}_1^T}, \underbrace{a_2, b_2, c_2, d_2}_{\mathbf{x}_2^T}, \cdot, \cdot; \underbrace{\alpha_1, \beta_1, \gamma_1, \delta_1}_{\mathbf{y}_1^T}, \underbrace{\alpha_2, \beta_2, \gamma_2, \delta_2}_{\mathbf{y}_2^T}, \cdot, \cdot]^T, \tag{7}$$

where $a_j = A_{II}\mathbf{x}_j, b_j = B_{II}\mathbf{x}_j, c_j = C_{II}\mathbf{x}_j, d_j = D_{II}\mathbf{x}_j$ and $\alpha_k = A_{II}\mathbf{y}_k, \beta_k = B_{II}\mathbf{y}_k, \gamma_k = C_{II}\mathbf{y}_k, \delta_k = D_{II}\mathbf{y}_k$. According to such a representation, given any factorized linear operator $W_I \otimes X_{II}$ of $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ we have

$$(W_I \otimes X_{II})\Psi = \begin{bmatrix} \sum_j \hat{p}_{1j} X_{II}\mathbf{x}_j + \sum_k \hat{u}_{1k} X_{II}\mathbf{y}_k \\ \sum_j \hat{p}_{2j} X_{II}\mathbf{x}_j + \sum_k \hat{u}_{2k} X_{II}\mathbf{y}_k \\ \cdot \\ \sum_j \hat{v}_{1j} X_{II}\mathbf{x}_j + \sum_k \hat{q}_{1k} X_{II}\mathbf{y}_k \\ \sum_j \hat{v}_{2j} X_{II}\mathbf{x}_j + \sum_k \hat{q}_{2k} X_{II}\mathbf{y}_k \\ \cdot \end{bmatrix}, \tag{8}$$

where $\hat{p}_{ij} = \langle f_i | W_I | f_j \rangle$, $\hat{u}_{ik} = \langle f_i | W_I | g_k \rangle$, $\hat{v}_{kj} = \langle g_k | W_I | f_j \rangle$ and $\hat{q}_{lk} = \langle g_l | W_I | g_k \rangle$. Then, in our representation $W_I \otimes X_{II}$ will be identified with the 'four blocks' matrix

$$\begin{bmatrix} \hat{p}_{11}X_{II} & \hat{p}_{12}X_{II} & \cdot & \hat{u}_{11}X_{II} & \hat{u}_{12}X_{II} & \cdot \\ \hat{p}_{21}X_{II} & \hat{p}_{22}X_{II} & \cdot & \hat{u}_{21}X_{II} & \hat{u}_{22}X_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hat{v}_{11}X_{II} & \hat{v}_{12}X_{II} & \cdot & \hat{q}_{11}X_{II} & \hat{q}_{12}X_{II} & \cdot \\ \hat{v}_{21}X_{II} & \hat{v}_{22}X_{II} & \cdot & \hat{q}_{21}X_{II} & \hat{q}_{22}X_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

so that $(W_I \otimes X_{II})\Psi$ in (8) turns out to be the classic matrix product of this matrix with the column vector (7).

In particular, we have

$$E = \begin{bmatrix} \mathbf{1}_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & \mathbf{1}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad T = \begin{bmatrix} T_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & T_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & T_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & T_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}; \tag{9i}$$

$$G = \begin{bmatrix} p_{11}\mathbf{1}_{II} & p_{12}\mathbf{1}_{II} & \cdot & u_{11}\mathbf{1}_{II} & u_{12}\mathbf{1}_{II} & \cdot \\ p_{21}\mathbf{1}_{II} & p_{22}\mathbf{1}_{II} & \cdot & u_{21}\mathbf{1}_{II} & u_{22}\mathbf{1}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{11}\mathbf{1}_{II} & v_{12}\mathbf{1}_{II} & \cdot & q_{11}\mathbf{1}_{II} & q_{12}\mathbf{1}_{II} & \cdot \\ v_{21}\mathbf{1}_{II} & v_{22}\mathbf{1}_{II} & \cdot & q_{21}\mathbf{1}_{II} & q_{22}\mathbf{1}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad Y = \begin{bmatrix} Y_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & Y_{II} & \cdot & \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & Y_{II} & \mathbf{0}_{II} & \cdot \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \cdot & \mathbf{0}_{II} & Y_{II} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad (9ii)$$

where $p_{ij} = \langle f_i|G_I|f_j\rangle$, $u_{ik} = \langle f_i|G_I|g_k\rangle$, $v_{kj} = \langle g_k|G_I|f_j\rangle$ and $q_{lk} = \langle g_l|G_I|g_k\rangle$.

By using (9.i) and (6), (7), from condition (C.3) $T\Psi = E\Psi$ we obtain $\mathbf{x}_j = [a_j, b_j, 0, 0]^T$, $\mathbf{y}_k = [0, 0, \gamma_k, \delta_k]^T$, i.e.

$$\Psi = [\underbrace{a_1, b_1, 0, 0}_{\mathbf{x}_1^T}, \underbrace{a_2, b_2, 0, 0}_{\mathbf{x}_2^T}, \cdot, \cdot; \underbrace{0, 0, \gamma_1, \delta_1}_{\mathbf{y}_1^T}, \underbrace{0, 0, \gamma_2, \delta_2}_{\mathbf{y}_2^T}, \cdot, \cdot]^T. \qquad \text{(gc.1)}$$

Then by using (10.ii) and (gc.1), we see that condition (C.4) $G\Psi = Y\Psi$ is equivalent to

$$\text{(i)} \begin{cases} \sum_i p_{ji}a_i = a_j \\ \sum_i p_{ji}b_i = 0, \end{cases} \quad \text{(ii)} \begin{cases} \sum_l u_{jl}\gamma_l = 0 \\ \sum_l u_{jl}\delta_l = 0, \end{cases}$$
$$\text{(iii)} \begin{cases} \sum_i v_{ki}a_i = 0 \\ \sum_i v_{ki}b_i = 0, \end{cases} \quad \text{(iv)} \begin{cases} \sum_l q_{kl}\gamma_l = \gamma_k \\ \sum_l q_{kl}\delta_l = 0. \end{cases} \qquad \text{(gc.2)}$$

Conditions (gc.1) and (gc.2) are general constraints to be satisfied in order that Ψ and G give rise to a solution of $(\mathcal{P})$. The following conditions (gc.3) and (gc.4) are straightforward consequences:

$$E\psi = T\Psi = [a_1, b_1, 0, 0, a_2, b_2, 0, 0, \cdots; 0, 0, 0, 0, 0, 0, 0, 0, \cdots]^T, \qquad \text{(gc.3)}$$
$$G\Psi = Y\Psi = [a_1, 0, 0, 0, a_2, 0, 0, 0, \cdots; 0, 0, \gamma_1, 0, 0, 0, \gamma_2, 0, \cdots]^T. \qquad \text{(gc.4)}$$

4.2. *dim* $(\mathcal{H}_I) = 4$*: correlated solutions*

We begin our search for solutions of $(\mathcal{P})$ by establishing that no solution exists if dim $(\mathcal{H}_I) = 2$. Indeed, in this case $G = \left[\begin{smallmatrix} p\mathbf{1}_{II} & u\mathbf{1}_{II} \\ v\mathbf{1}_{II} & q\mathbf{1}_{II} \end{smallmatrix}\right]$, where p, u, v, q are complex numbers, with $u = \bar{v} \neq 0$ to satisfy $[G, E] \neq \mathbf{0}$, and $\Psi = [a, b, 0, 0; 0, 0, \gamma, \delta]$. Then (gc.2.ii) and (gc.2.iii) respectively imply $\gamma = \delta = 0$ and $a = b = 0$, i.e. $\Psi = 0$.

In considering higher dimensions, we shall restrict ourselves to the case that the slits are symmetrical, which implies to exclude odd dimensions. In [24] we gave a solution of $(\mathcal{P})$ with dim $(\mathcal{H}_{II}) = 2$ and dim $(\mathcal{H}_I) = 4$. However, for this solution the outcomes of the two detections of E and G coincide. Now we prove that this trivial character is shared by every solution of $(\mathcal{P})$, if dim $(\mathcal{H}_I) = 4$, independent of the dimension of $\mathcal{H}_{II}$. In this case, the representation (10) of the projection operators E, G, T and Y in a solution of $(\mathcal{P})$ is made of 4×4 matrices, whereas Ψ in (gc.1) is

$$\Psi = [a_1, b_1, 0, 0, a_2, b_2, 0, 0; 0, 0, \gamma_1, \delta_1, 0, 0, \gamma_2, \delta_2]^T.$$

As a consequence, conditions (gc.2) become $(j = 1, 2)$

$$\text{(i)} \begin{cases} p_{j1}a_1 + p_{j2}a_2 = a_j \\ p_{j1}b_1 + p_{j2}b_2 = 0, \end{cases} \quad \text{(ii)} \begin{cases} u_{j1}\gamma_1 + u_{j2}\gamma_2 = 0 \\ u_{j1}\delta_1 + u_{j2}\delta_2 = 0, \end{cases}$$
$$\text{(iii)} \begin{cases} v_{j1}a_1 + v_{j2}a_2 = 0 \\ v_{j1}b_1 + v_{j2}b_2 = 0, \end{cases} \quad \text{(iv)} \begin{cases} q_{j1}\gamma_1 + q_{j2}\gamma_2 = \gamma_j \\ q_{j1}\delta_2 + q_{j2}\delta_2 = 0. \end{cases} \qquad (10)$$

In order that $[G, E] \neq \mathbf{0}$, at least one of the entries u_{ij} must be different from 0. This implies that the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ must be linearly dependent. Let us suppose that $\gamma_2 = \lambda\gamma_1$ and $\delta_2 = \lambda\delta_1$. By using these relations in (11.iv), we get

$$\begin{cases} q_{11}\gamma_1 + q_{12}\gamma_2 = \gamma_1 = (q_{11} + \lambda q_{12})\gamma_1 \\ q_{21}\gamma_1 + q_{22}\gamma_2 = \gamma_2 = (q_{21} + \lambda q_{22})\gamma_1 \\ q_{11}\delta_1 + q_{12}\delta_2 = 0 = (q_{11} + \lambda q_{12})\delta_1 \\ q_{21}\delta_1 + q_{22}\delta_2 = 0 = (q_{21} + \lambda q_{22})\delta_1, \end{cases}$$

If $\delta_1 \neq 0$ then $(q_{11}+\lambda q_{12}) = (q_{21}+\lambda q_{22}) = 0$, which implies $\gamma_1 = \gamma_2 = 0$. On the other hand, if $\delta_1 = 0$ then $\delta_2 = \lambda\delta_1 = 0$, while γ_1, γ_2 can be non-vanishing with $\gamma_2 = \lambda\gamma_1$. Similarly, we can show that if $b_1 \neq 0$ then $a_1 = a_2 = 0$; if $b_1 = 0$ then $b_2 = 0$ and $a_2 = \mu a_1$. Hence, the following statements are implied by (11):

(a) if $b_1 = 0$ and $\delta_1 = 0$, then
$\Psi = [a_1, 0, 0, 0, \mu a_1, 0, 0, 0; 0, 0, \gamma_1, 0, 0, 0, \lambda\gamma_1, 0]^T$;
(b) if $b_1 = 0$ and $\delta_1 \neq 0$, then
$\Psi = [a_1, 0, 0, 0, \mu a_1, 0, 0, 0; 0, 0, 0, \delta_1, 0, 0, 0, \lambda\delta_1]^T$;
(c) if $b_1 \neq 0$ and $\delta_1 = 0$, then
$\Psi = [0, b_1, 0, 0, 0, \mu b_1, 0, 0; 0, 0, \gamma_1, 0, 0, 0, \lambda\gamma_1, 0]^T$;
(d) if $b_1 \neq 0$ and $\delta_1 \neq 0$, then
$\Psi = [0, b_1, 0, 0, 0, \mu b_1, 0, 0; 0, 0, 0, \delta_1, 0, 0, 0, \lambda\delta_1]^T$.

Cases (a) and (d) violate (C.5) because they respectively yield $G\Psi = \Psi$ and $G\Psi = 0$. Therefore, a state vector in a solution of $(\mathcal{P})$ must have one of the forms in (b), (c).

If case (b) (respectively case (c)) for Ψ is realized then, taking into account (gc.3) and (gc.4), we get $T\Psi = Y\Psi$ (respectively $(\mathbf{1}_{II} - T)\Psi = Y\Psi$). This is equivalent to stating that the conditional probability $P(T|Y)$ (respectively $P(T|Y')$) is equal to 1. As a consequence, we can conclude that *property G is detected by Y on a particle (i.e. the outcome for Y is 1) if and only if T detects the passage of that particle through slit 1 (respectively slit 2).* Thus, we have perfect correlation.

4.3. Non-correlated solutions

In this subsection we answer the question whether, by allowing the dimension of $\mathcal{H}_I$ to be at least 6, solutions of $(\mathcal{P})$ exist, without correlations between the detections of G and E, always present in the (even) cases dim $(\mathcal{H}_I) < 6$. We assume that $\text{rank}(E_I) = \text{rank}(\mathbf{1}_I - E_I) = 3$, so that $\Psi = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]^T$ and in (gc.2) indices i, j, k, l take values in $\{1, 2, 3\}$. Now we show that in order to have non-correlated solutions everyone of the triples $\{\delta_1, \delta_2, \delta_3\}$ and $\{b_1, b_2, b_3\}$ must be generated by just one vector, δ and b respectively.

Since at least one of the entries u_{ij} must be non-zero to satisfy $[G, E] \neq \mathbf{0}$, general constraint (gc.2.ii) implies that one of the three vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$—say $\mathbf{y}_1$—is a linear combination of the remaining two:

$$\begin{cases} \delta_1 = \lambda_{12}\delta_2 + \lambda_{13}\delta_3 \\ \gamma_1 = \lambda_{12}\gamma_2 + \lambda_{13}\gamma_3. \end{cases} \tag{11}$$

Using these equations in (gc.2.iv), we get

$$\begin{cases} (q_{k2} + \lambda_{12}q_{k1})\delta_2 + (q_{k3} + \lambda_{13}q_{k1})\delta_3 = 0 \\ \gamma_k = (q_{k2} + \lambda_{12}q_{k1})\gamma_2 + (q_{k3} + \lambda_{13}q_{k1})\gamma_3, \end{cases} \qquad k = 1, 2, 3. \tag{12}$$

If the vectors δ_2, δ_3 are linearly independent, then the first equation in (12) implies $(q_{k2} + \lambda_{12} q_{k1}) = (q_{k3} + \lambda_{13} q_{k1}) = 0$, so that the second equation in (12) yields $\gamma_k = 0$ for all k. Hence,

$$\delta_2, \delta_3 \text{ linearly independent} \quad \Rightarrow \quad \mathbf{y}_k = [0, 0, 0, \delta_k]^T, \forall k. \tag{a.i}$$

In a similar way, we can prove that

$$b_2, b_3 \text{ linearly independent} \quad \Rightarrow \quad \mathbf{x}_k = [0, b_k, 0, 0]^T, \forall k. \tag{a.ii}$$

Now we draw the consequences of (a.i)–(a.ii) relative to our problem $(\mathcal{P})$. Given a state vector Ψ satisfying general constraint (gc.1), a possibility is that

(a) δ_2, δ_3 are linearly independent and b_2, b_3 are also linearly independent.
In this case $\mathbf{x}_k = [0, b_k, 0, 0]^T$ and $\mathbf{y}_k = [0, 0, 0, \delta_k]^T$. If a solution of $(\mathcal{P})$ exists, then $G\Psi = Y\Psi = 0$ would follow from (gc.4), and condition (C.5) would be violated. If we consider the other cases, then we obtain the following implications.

(b) δ_2, δ_3 linearly independent and b_2, b_3 linearly dependent imply
$\mathbf{x}_k = [a_k, b_k, 0, 0]^T$ and $\mathbf{y}_k = [0, 0, 0, \delta_k]^T$.

(c) δ_2, δ_3 linearly dependent and b_2, b_3 linearly independent imply
$\mathbf{x}_k = [0, b_k, 0, 0]^T$ and $\mathbf{y}_k = [0, 0, \gamma_k, \delta_k]^T$.

(d) δ_2, δ_3 linearly dependent and b_1, b_2 linearly dependent imply
$\mathbf{x}_k = [a_k, b_k, 0, 0]^T$ and $\mathbf{y}_k = [0, 0, \gamma_k, \delta_k]^T$.

Now we can see that only case (d) leads to non-correlated solutions. In case (b), if a solution of $(\mathcal{P})$ exists such that δ_2, δ_3 are linearly independent and b_2, b_3 are linearly dependent, then (gc.3), (gc.4) imply that $YT\Psi = Y\Psi$ holds, which is equivalent to saying that the conditional probability $p(T|Y) = \langle\Psi|TY\Psi\rangle/\langle\Psi|Y\Psi\rangle$ is equal to 1; this means that each time a particle is measured to have $T = 1$, then it certainly has $Y = 1$. In case (c) $TY\Psi = T\Psi$ holds, so that each time a particle is sorted by Y, then it is certainly sorted by T. Therefore, for all eventual solutions corresponding to cases (b) and (c), property G must be correlated with the WS property E.

Hence, to concretely find non-correlated solutions, we have to take state vectors Ψ such that the triples $\delta_1, \delta_2, \delta_3$ and b_1, b_2, b_3 are generated by just one vector, δ and b respectively. Now we look for particular solutions corresponding to state vectors Ψ such that

$$\begin{cases} a_1 = a_2 = 0, a_3 \neq 0, b_2 = \mu b_1 \neq 0, b_3 = 0, \\ \gamma_1 = \gamma_2 = 0, \gamma_3 \neq 0, \delta_2 = \lambda\delta_1 \neq 0, \delta_3 = 0. \end{cases} \tag{13}$$

Then (gc.2.i) and (gc.2.iv) respectively imply $p_{13} = p_{23} = 0, p_{33} = 1$ and $q_{13} = q_{23} = 0, q_{33} = 1$. Similarly, (gc2.iii) implies $v_{13} = v_{23} = v_{33} = 0$ and hence, by the self-adjointness of G_I, $u_{31} = u_{32} = u_{33} = 0$. On the other hand, the first equation in (gc.2.ii) and (13) imply $u_{13} = u_{23} = 0$ and hence $v_{31} = v_{33} = 0$. Therefore, taking into account (13), conditions (gc.2) become

$$\begin{array}{llll} \text{(i)} & \begin{cases} p_{11} + \mu p_{12} = 0 \\ p_{21} + \mu p_{22} = 0, \end{cases} & \text{(ii)} & \begin{cases} u_{11} + \lambda u_{12} = 0 \\ u_{21} + \lambda u_{22} = 0, \end{cases} \\ \text{(iii)} & \begin{cases} v_{11} + \mu v_{12} = 0 \\ v_{21} + \mu v_{22} = 0, \end{cases} & \text{(iv)} & \begin{cases} q_{11} + \lambda q_{12} = 0 \\ q_{21} + \lambda q_{22} = 0. \end{cases} \end{array} \tag{14}$$

The self-adjointness of G, together with (14), yields

$$G = \begin{bmatrix} p\mathbf{1}_{II} & -p/\mu\mathbf{1}_{II} & \mathbf{0}_{II} & u\mathbf{1}_{II} & -u/\lambda\mathbf{1}_{II} & \mathbf{0}_{II} \\ -p/\bar{\mu}\mathbf{1}_{II} & p/|\mu|^2\mathbf{1}_{II} & \mathbf{0}_{II} & -u/\bar{\mu}\mathbf{1}_{II} & u/\lambda\bar{\mu}\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} \\ \bar{u}\mathbf{1}_{II} & -\bar{u}/\mu\mathbf{1}_{II} & \mathbf{0}_{II} & q\mathbf{1}_{II} & -q/\lambda\mathbf{1}_{II} & \mathbf{0}_{II} \\ -\bar{u}/\bar{\lambda}\mathbf{1}_{II} & \bar{u}/\bar{\lambda}\mu\mathbf{1}_{II} & \mathbf{0}_{II} & -q/\bar{\lambda}\mathbf{1}_{II} & q/|\lambda|^2\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} \end{bmatrix}, \tag{15}$$

where we have put $p = p_{11}$, $u = u_{11}$, $v = v_{11}$, $q = q_{11}$. By imposing idempotence, we find that in correspondence with $\lambda = \mu = 1$, we have the following solution for G:

$$G = \begin{bmatrix} p\mathbf{1}_{II} & -p\mathbf{1}_{II} & \mathbf{0}_{II} & \mathrm{e}^{\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & -\mathrm{e}^{\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathbf{0}_{II} \\ -p\mathbf{1}_{II} & p\mathbf{1}_{II} & \mathbf{0}_{II} & -\mathrm{e}^{\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathrm{e}^{\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} \\ \mathrm{e}^{-\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & -\mathrm{e}^{-\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathbf{0}_{II} & \left(\frac{1}{2}-p\right)\mathbf{1}_{II} & -\left(\frac{1}{2}-p\right)\mathbf{1}_{II} & \mathbf{0}_{II} \\ -\mathrm{e}^{-\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathrm{e}^{-\mathrm{i}\theta}\sqrt{p\left(\frac{1}{2}-p\right)}\mathbf{1}_{II} & \mathbf{0}_{II} & -\left(\frac{1}{2}-p\right)\mathbf{1}_{II} & \left(\frac{1}{2}-p\right)\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} \end{bmatrix} \tag{16}$$

such that $\mathrm{rank}(G_I) = 3$, for every p such that $0 < p < 1/2$ and any $\theta \in \mathbf{R}$.

For instance, for $\theta = 0$ and $p = 1/4$, taking into account (13), we get the following solution of $(\mathcal{P})$:

$$\Psi = [0, b_1, 0, 0, 0, b_1, 0, 0, a_3, 0, 0, 0; 0, 0, 0, \delta_1, 0, 0, 0, \delta_1, 0, 0, \gamma_3, 0]^T, \tag{17i}$$

$$G = \begin{bmatrix} \frac{1}{4}\mathbf{1}_{II} & -\frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} & \frac{1}{4}\mathbf{1}_{II} & -\frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} \\ -\frac{1}{4}\mathbf{1}_{II} & \frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} & -\frac{1}{4}\mathbf{1}_{II} & \frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} \\ \frac{1}{4}\mathbf{1}_{II} & -\frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} & \frac{1}{4}\mathbf{1}_{II} & -\frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} \\ -\frac{1}{4}\mathbf{1}_{II} & \frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} & -\frac{1}{4}\mathbf{1}_{II}\mathbf{1}_{II} & \frac{1}{4}\mathbf{1}_{II} & \mathbf{0}_{II} \\ \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{0}_{II} & \mathbf{1}_{II} \end{bmatrix}. \tag{17ii}$$

The ideal experiment of section 3.1 corresponds to the particular solution of this kind where $b_1 = (\sqrt{3}/4)|1/2\rangle$, $a_3 = (1/\sqrt{8})|3/2\rangle$, $\delta_1 = (1/4)|-3/2\rangle$, $\gamma_3 = \sqrt{3/8}|-1/2\rangle$.

5. Conclusive remarks

As a particular quantum measurement, our detection scheme inherits the so-called *measurement problem* [25] from quantum mechanics. To see this, we make use of a simplified argument, where $\mathcal{H}_{II}$ is identified with the detector space. In a solution of problem $(\mathcal{P})$ with $\dim(\mathcal{H}_{II}) = 4$, the simultaneous detection of E and G is made possible by the entanglement, encoded in the state vector (17.i) which we may rewrite as $\Psi = \phi_1|1\rangle + \phi_2|2\rangle + \phi_3|3\rangle + \phi_4|4\rangle$, between the vectors ϕ_k of $\mathcal{H}_I$ and the alternative *pointer* states $|1\rangle \propto a_3$, $|2\rangle \propto b_1$, $|3\rangle \propto \gamma_3$, $|4\rangle \propto \delta_1$ of the detector space $\mathcal{H}_{II}$, but we could arbitrarily choose another pointer basis $\{|\xi\rangle, |\eta\rangle, |\tau\rangle, |\zeta\rangle\} \in \mathcal{H}_{II}$, so that the same state vector Ψ can be written as $\Psi = \varphi_1|\xi\rangle + \varphi_2|\eta\rangle + \varphi_3|\tau\rangle + \varphi_4|\zeta\rangle$. Hence, the same Ψ encodes entanglement with another pointer, so that an alternative pair of non-commuting properties, say $\hat{E}$ and $\hat{G}$, could be detectable together with the measurement of the final impact point; therefore, the natural conclusion that a definite outcome has been objectively realized for the detection of E and G, i.e. the system decoheres [14], cannot be drawn, not even in the case that the detector was prepared in advance just to detect E and G. According to quantum theory, such a conclusion could be correctly stated only if the density operator associated with the quantum state is of the *reduced* form

$$\rho^r = p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + p_3|3\rangle\langle 3| + p_4|4\rangle\langle 4|, \tag{18}$$

whereas the actual density operator ρ_Ψ corresponding to Ψ is the 'Schroedinger cat' state $\rho^\Psi = |\Psi\rangle\langle\Psi| = \rho^r + \sum_{j\neq k}\lambda_{j,k}|j\rangle\langle k|$. This is the form of the measurement problem of quantum mechanics in our specific framework. Now we briefly outline how the theory of 'environment-induced decoherence', one of the approaches developed to solve this fundamental problem [14, 26], can also apply in the present case. Following this approach, the whole system composed of the localizable particle plus the detector cannot be treated as a closed system, because a certain amount of interaction with the rest of the universe (the *environment* $\mathcal{E}$ described in a Hilbert space $\mathcal{H}_\mathcal{E}$) always exists. Then, a right Hilbert space to describe the detection process is $\mathcal{H}_I \otimes \mathcal{H}_{II} \otimes \mathcal{H}_\mathcal{E}$. The presence of the apparatuses prepared for detecting E and G forces the dynamics to also involve $\mathcal{E}$, in such a way that the right state vector is $\hat{\Psi} = \phi_1|1\rangle|\epsilon_1\rangle + \phi_2|2\rangle|\epsilon_2\rangle + \phi_3|3\rangle|\epsilon_3\rangle + \phi_4|4\rangle|\epsilon_4\rangle$, $(|\epsilon_k\rangle \in \mathcal{H}_\mathcal{E})$ which corresponds to the density operator $\hat{\rho} = |\hat{\Psi}\rangle\langle\hat{\Psi}|$. The state of our open sub-system of the entire universe, described in $\mathcal{H}_I \otimes \mathcal{H}_{II}$, can be univocally obtained by tracing on any basis of $\mathcal{H}_\mathcal{E}$: $\rho_{I+II} = \sum_{\epsilon_k}\langle\epsilon_k|\hat{\rho}|\epsilon_k\rangle$ [23]. In so doing it turns out that $\rho_{I+II} = \sum_k p_k|k\rangle\langle k| = \rho^r$, which is the right form (18) for avoiding the measurement problem.

The results of the present work provide interesting insights also with regard to the consistent histories theory. Different from environmental decoherence, where the decoherence is a dynamical consequence of the interaction with the environment $\mathcal{E}$, the consistent histories approach (CHA) is an extension of standard quantum theory aimed to describe how a quantum system can decohere as an isolated system [15–17]. To this aim the CHA introduces the concept of *consistent family of histories*, i.e. a suitable family of alternative sequences $h = (E_1, E_2, \ldots, E_n)$, said histories, of quantum properties (projections E_k in h) the system can objectively possess at the fixed respective times t_1, t_2, ..., t_n during its unitary evolution as an isolated system, ruled over by the Schroedinger equation; this entails that we can consider histories of a consistent family exactly like events of a classical sample space. Thereby [16, 27], the condition Re $(Tr(C_{h_1}\rho C^*_{h_2})) = 0$ for all mutually exclusive histories h_1, h_2 can be implied, where $C_h = E_n \cdot E_{n-1} \cdots E_1$. This last mathematical condition, known as *weak decoherence*, is the criterion adopted by Griffiths to select (consistent) families of decohering histories. Other authors argue that stronger conditions are necessary [17].

In the present theory of double detection, modeled by means of two-slits experiment, we argue of attributing properties E and/or G at time t_1 and properties $F(\Delta)$ at time t_2: in fact, we are treating with two-times histories. Namely, two families of histories naturally arise: the family $\mathcal{C}^E$ of histories of the kind $(E, F(\Delta))$ and the family $\mathcal{C}^G$ of histories of the kind $(G, F(\Delta))$. It is not difficult to single out relationships between the detectability of E or G as treated in section 4, and the consistency of the corresponding family, by making use of a result proved in [28]. In that paper it is proved (Proposition 2) that the existence of a detector T of E (respectively Y of G) implies that family $\mathcal{C}^E$ (respectively $\mathcal{C}^G$) is weakly decoherent. Therefore, we can easily derive the following implication.

($\mathcal{R}$) *If* Ψ *is a vector state in a solution of problem* ($\mathcal{P}$), *then both families* $\mathcal{C}^E$ and $\mathcal{C}^G$ *are weakly decoherent.*

It would seem quite natural to interpret the detection of E (the occurrence of outcome 1 for T) and the measured occurrence of $F(\Delta)$ as the occurrence of history $h_1 = (E, F(\Delta))$, in the sense of the CHA. Now we show how this interpretation is inconsistent with the CHA notion of consistent family. Since $[E, G] \neq \mathbf{0}$, the families $\mathcal{C}^E$ and $\mathcal{C}^G$ are *incompatible* families: it can be shown that no consistent family $\mathcal{C}$ exists which contains both $\mathcal{C}^E$ and $\mathcal{C}^G$ [15, 16]; therefore, no individual specimen of the physical system can simultaneously follow history $h_1 = (E, F(\Delta)) \in \mathcal{C}^E$ and $h_2 = (G, F(\Delta)) \in \mathcal{C}^G$. But in our solution, for instance that of section 3.1, the detections of E and G do occur simultaneously for the same specimen; then we should conclude that both histories h_1 and h_2 have occurred, whenever outcome 1 occur for both T and Y.

Thus, the existence of non-trivial solutions of problem ($\mathcal{P}$), proved in the present work, gives rise to an interesting interpretative issue in connection with the CHA, perhaps related to the debated issue (see [29] and references therein) of the choice of a single consistent family.

References

[1] Institut International de Physique Solvay, Rapport et discussions du 5^e Conseil, Paris 1928

[2] Bohr N 1983 *Quantum Theory of Measurement* ed J A Wheeler and W H Zurek (Princeton, NJ: Princeton University Press) p 9

[3] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)

[4] Lahti P J 1983 *Int. J. Theor. Phys.* **22** 911

[5] Von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)

[6] Bohr N 1949 *Albert Einstein: Philosopher-Scientist* ed P A Schilpp (Evanston: Library of Living Philosophers p 200)

[7] Feynman R, Leighton R and Sands M 1965 *The Feynman Lectures on Physics, III* (Reading, MA: Addison-Wesley)

[8] Scully M O, Englert B-G and Walther H 1991 *Nature* **351** 111

[9] Scully M O and Walther H 1989 *Phys. Rev.* A **39** 5229

[10] Englert B-G, Schwinger J and Scully M O 1990 *New Frontiers in Quantum Electrodynamics and Quantum Optics* ed A O Barut (New York: Plenum) p 507

[11] Wooters W K and Zurek W H 1987 *Phys. Rev.* D **19** 473

[12] Jaeger G, Shimony A and Vaidman L 1995 *Phys. Rev.* A **51** 54

[13] Englert B-G and Bergou J A 2000 *Opt. Commun.* **179** 337

[14] Zurek W H 1991 *Phys. Today* **44** 36

[15] Griffiths R B 1984 *J. Stat. Phys.* **36** 219

[16] Omnès R 1999 *Understanding Quantum Mechanics* (Princeton, NJ: Princeton University Press)

[17] Gell-Mann M and Hartle J B 1993 *Phys. Rev.* D **47** 3345

[18] Nisticò G and Romania M C 1994 *J. Math. Phys.* **35** 4534

[19] Vaidman L, Aharonov Y and Albert D Z 1987 *Phys. Rev. Lett.* **58** 1385

[20] Schwindt P D D, Kwiat P G and Englert B-G 1999 *Phys. Rev.* A **60** 4285

[21] Durr S, Nonn T and Rempe G 1998 *Nature* **395** 33

[22] Kunze S, Durr S and Rempe G 1996 *Europhys. Lett.* **34** 343

[23] Jauch J M 1968 *Foundations of Quantum Mechanics* (Reading, MA: Addison-Wesley)

[24] Nisticò G and Sestito A 2004 *J. Mod. Opt.* **51** 1063

[25] Wheeler J A and Zurek W H (ed) 1983 *Quantum Theory and Measurements* (Princeton, NJ: Princeton University Press)

[26] Cucchietti F M, Paz J H and Zurek W H 2005 *Phys. Rev.* A **72** 052113

[27] Nisticò G 1999 *Found. Phys.* **29** 221

[28] Nisticò G 2002 *Phys. Lett.* A **299** 433

[29] Griffiths R B 1998 *Phys. Rev.* A **57** 1604